

# Non-equilibrium quasi-condensates in reduced dimensions

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**Abstract** –We develop a generic theory of the fluctuations on top of a non-equilibrium Bose-Einstein condensate in reduced dimensions. Analytic expressions for the momentum distribution of the non-condensed cloud and for the long-distance behavior of the spatial coherence are obtained for the different cases. Comparison of our predictions with available experimental data on condensates of exciton-polaritons and on surface-emitting planar laser devices is finally made.

Non-equilibrium phase transitions are among the most exciting topics of non-equilibrium statistical mechanics [1]. In the last decades they have been studied in a variety of different physical systems, of either classical [2–4] or quantum [5, 6] nature. However, as compared to their equilibrium counterparts, much less is known about their general features, in particular for what concerns the critical behavior in the vicinity of the transition point. A most important class of non-equilibrium phase transitions take place in systems at the so-called non-equilibrium steady state (NESS) where a dynamical balance of driving and dissipation replaces the usual thermal equilibrium condition of standard equilibrium statistical mechanics.

Even though laser operation has played a central role in most developments of contemporary experimental atomic, molecular and optical physics, its potential as a workbench for non-equilibrium statistical mechanics studies has not been fully exploited yet. The interpretation of the laser threshold as a second-order phase transition dates back to the early 1970's with the pioneering works by DeGior- gio and Scully [7] and by Graham and Haken [8, 9]: the order parameter of the transition is the amplitude  $E$  of the electromagnetic field in the laser cavity, which gets a well defined phase by spontaneously breaking the  $U(1)$  symmetry corresponding to phase rotations,  $E \rightarrow E e^{i\theta}$ .

While most textbook discuss this analogy in the case of single-mode cavities only, the spontaneous symmetry breaking phenomenon is -rigorously speaking- restricted to spatially extended systems. Only in this case, it is in

fact meaningful to take the long-distance limit involved in the Penrose-Onsager definition of an ordered state,

$$\lim_{|\mathbf{x}-\mathbf{x}'| \rightarrow \infty} \langle \hat{E}^\dagger(\mathbf{x}) \hat{E}(\mathbf{x}') \rangle \neq 0. \quad (1)$$

Most remarkable examples of spatially extended laser devices are the so-called vertical cavity surface emitting lasers (VCSELs), whose planar geometry makes a continuum of in-plane modes to be available to both lasing and fluctuations. Even though such VCSEL devices have attracted a strong interest from the point of view of non-linear optics and of all-optical information processing [10], very little work has been devoted to the non-equilibrium statistical mechanics features of their operation. On the other hand, a great attention has been devoted in the last years to studies of the closely related Bose-Einstein condensation (BEC) phase transition in gases of exciton-polaritons in semiconductor microcavities [11]. In the strong light-matter coupling regime, the cavity photon is strongly mixed to an excitonic transition in the cavity medium, giving rise to new long-lived bosonic excitation modes, the so-called exciton-polaritons: for strong enough pump intensities, the polariton density reaches the threshold for BEC and long-range spatial and temporal coherence appears. Recently, BEC was also observed in a photon gas in a macroscopic cavity filled of dye molecules [12].

Even though most features of these BEC experiments are accurately described under the assumption that thermal equilibrium is reached within the photon/polariton gas on a time-scale shorter than the particle decay rate

$\gamma$ , several authors have pointed out intriguing consequences of the non-equilibrium nature of these optical BECs, whose steady-state is determined by a dynamical balance of pumping and particle losses: the Goldstone mode associated to the spontaneously broken  $U(1)$  symmetry has a diffusive rather than sonic nature [13–15] and the momentum-space shape of the condensate strongly depends on the pump spot geometry [16, 17].

As the spontaneously broken symmetry is the same  $U(1)$  in all cases of polariton BEC, photon BEC, and laser operation, it is possible to construct a unified theoretical description of these phenomena under the general name of *non-equilibrium Bose-Einstein condensation*: the role of stimulated emission of photons by a population-inverted laser medium is played in the BEC case by stimulated scattering from the thermal cloud into the condensate mode; the actual distance from an equilibrium state may of course depend on the microscopic details of the system. The purpose of this Letter is to develop such a general theory and then to apply it to the long-range coherence properties of (quasi)-condensates in reduced dimensions: independently of collisional thermalization processes, we shall see that fluctuations in the non-equilibrium steady-state recover the ones of the corresponding equilibrium system at a temperature determined by the characteristic decay rate.

**Stochastic Gross-Pitaevskii Equation.** – Among the different theoretical frameworks have been used to study non-equilibrium BEC transitions in planar optical devices [14, 18, 19], in this Letter we shall use the simple, yet powerful techniques based on the so-called stochastic Gross-Pitaevskii equation (SGPE). The idea is to start from the mean-field equation for the coherent condensate field and then to add a stochastic noise term to include quantum and/or classical fluctuations. For the case of a coherent pump, this procedure can be made rigorous within the Wigner representation of the quantum field and has been successfully applied in Monte-Carlo studies of the field coherence across the transition point [13, 20]. A first successful extension of this method to the incoherent pump case that is under investigation here was made in [21] on phenomenological grounds: a more rigorous derivation requires a quantum description of the amplifying mechanism that will be published in a future, more extensive work based on [22].

Independently of the microscopic nature of the Bose field  $\phi(\mathbf{x}, t)$  (which can describe equally well the electromagnetic field of a VCSEL and the polariton field of a polariton condensate), its low-energy physics can be accurately described by a  $d$ -dimensional SGPE of the form [15]:

$$i d\phi = \left[ \omega_0 - \frac{\hbar \nabla^2}{2m} + g|\phi|^2 + i \left( \frac{P_0}{1 + \frac{|\phi|^2}{n_s}} - \gamma \right) \right] \phi dt + dW, \quad (2)$$

where  $m$  is the boson mass of the field,  $\gamma$  is the damping rate,  $P_0$  is the pumping rate and  $n_s$  is the saturation

density of the pumping mechanism. As usual, the validity of the SGPE model requires that the nonlinear coupling strength  $g \geq 0$  describing repulsive contact interactions between the particles be weak enough for the gas to be dilute.

As usual in open systems, the presence of dissipative terms in the motion equations requires including corresponding stochastic terms  $dW(\mathbf{x}, t)$ . For the sake of simplicity, we phenomenologically choose the simplest Ito form of a temporally white complex Gaussian noise term with a random phase and spatially local correlations,

$$\langle dW(\mathbf{x}, t) dW^*(\mathbf{x}', t) \rangle = 2D_{\phi\phi} \delta^{(d)}(\mathbf{x} - \mathbf{x}') dt, \quad (3)$$

$$\langle dW(\mathbf{x}, t) dW(\mathbf{x}', t) \rangle = 0. \quad (4)$$

The assumption of a white, temporally uncorrelated noise is based on the physical assumption that the characteristic time of the bath is much shorter than the characteristic time of the system. A brief account of effects beyond the white bath approximation will be given in the last section of the Letter. An *ab initio* calculation of the diffusion coefficient  $D_{\phi\phi}$  will be presented in a future publication: based on previous work on the OPO pump case [20], we expect a value  $D_{\phi\phi} \approx \gamma$  for pump strengths ranging from zero to not too much above the transition point.

**Mean-field approximation and collective excitations.** – The first step to understand the behavior of the system is to perform a mean-field approximation and look for the stationary state of the deterministic GPE that is obtained by neglecting the noise terms in (2). For simplicity, we restrict our attention to the case of a spatially homogeneous system, for which we can look for a translationally symmetric mean field solution of the form  $\phi(\mathbf{x}, t) = \phi_0 e^{-i\omega t}$ .

Independently of the value of the pump intensity  $P_0$ , the trivial field  $\phi = 0$  is always a solution of the GPE. For low pump intensities below the threshold  $P_0 < \gamma$ , this is the only solution and is dynamically stable. On the other hand, for pump intensities  $P_0 > \gamma$  above the threshold, it becomes dynamically unstable and is replaced by another solution with a finite density  $|\phi_0|^2 = n_s(P_0 - \gamma)/\gamma$  and an oscillating frequency  $\omega = \omega_0 + g|\phi_0|^2$ : the global phase of  $\phi_0$  acquires a well-defined value by spontaneously breaking the  $U(1)$  symmetry of (2). In the analogy with second order phase transitions, the order parameter hallmarked the transition is then the complex field amplitude  $\phi_0$ .

The linearization of (2) around the finite solution determines a pair of coupled Bogoliubov partial differential equations for the fluctuation field  $\delta\phi(\mathbf{x}, t)$  defined as  $\phi(\mathbf{x}, t) = [\phi_0 + \delta\phi(\mathbf{x}, t)] e^{-i\omega t}$ , and its complex conjugate  $\delta\phi^*(\mathbf{x}, t)$ . These fluctuations describe the excitations of the Bose field on top of a pure condensate solution, that is the non-condensed fraction of the gas.

In the spatially homogeneous case under investigation here, it is useful to work in Fourier space and separate the evolution of the different in-plane components. For each

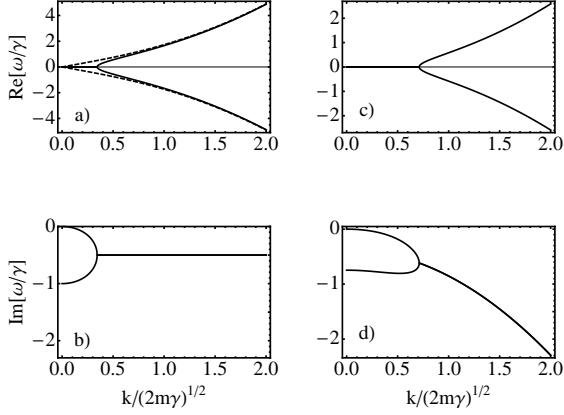


Fig. 1: Bogoliubov dispersion of collective excitations of the non-equilibrium condensate as a function of the (real) in-plane momentum  $\mathbf{k}$ . Upper panels show the real part of the frequency, lower panels show the imaginary part. The panels (a-b) in the left column are for the case of a frequency-independent bath. The panels (c-d) in the right column model the frequency-dependence of the bath as discussed in the last part of the Letter. System parameters:  $P_0 = 2\gamma$ ,  $gn_s = \gamma$ ,  $\omega_c = 2\gamma$ ,  $\omega_0 = 0$ .

(real-valued)  $\mathbf{k}$  we are then left with a pair of coupled linear differential equation for  $\delta\tilde{\phi}(\mathbf{k})$  and  $[\delta\tilde{\phi}(-\mathbf{k})]^*$ ,

$$i \frac{d}{dt} \begin{pmatrix} \delta\tilde{\phi}(\mathbf{k}) \\ (\delta\tilde{\phi}(-\mathbf{k}))^* \end{pmatrix} = \mathcal{L}_{\mathbf{k}} \begin{pmatrix} \delta\tilde{\phi}(\mathbf{k}) \\ (\delta\tilde{\phi}(-\mathbf{k}))^* \end{pmatrix} \quad (5)$$

with the  $\mathbf{k}$ -dependent Bogoliubov matrix  $\mathcal{L}_{\mathbf{k}}$  defined as

$$\mathcal{L}_{\mathbf{k}} = \begin{pmatrix} \epsilon_{\mathbf{k}} + \mu - i\Gamma & \mu - i\Gamma \\ -\mu - i\Gamma & -\epsilon_{\mathbf{k}} - \mu - i\Gamma \end{pmatrix} \quad (6)$$

in terms of the free particle dispersion  $\epsilon_{\mathbf{k}} = \hbar k^2/2m$  and the interaction energy  $\mu = g|\phi_0|^2$ . By simple diagonalization of  $\mathcal{L}_{\mathbf{k}}$ , one gets a double-branched excitation spectrum

$$\hbar\omega_{\mathbf{k}}^{\pm} = -i\hbar\Gamma \pm \hbar\sqrt{E_{\mathbf{k}}^2 - \Gamma^2}, \quad (7)$$

where  $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2\mu)}$  is the standard Bogoliubov dispersion of equilibrium systems and the effective damping rate  $\Gamma = \gamma(P_0 - \gamma)/P_0$ . An example of this non-equilibrium Bogoliubov dispersion is shown in the left panels of Fig.1: as first discussed in [13–15] its most remarkable feature is the diffusive nature of the Goldstone mode describing long-wavelength phase twists in the condensate: the real part of its frequency  $\omega_{\mathbf{k}}^+$  vanishes within a finite region of in-plane wave vectors around  $\mathbf{k} = 0$ , while its imaginary part grows quadratically from 0,  $\omega_{\mathbf{k}}^+ \simeq -i\hbar\mu k^2/m\Gamma$ . The other branch  $\omega_{\mathbf{k}}^-$  describing density fluctuations (intensity fluctuations in the language of laser physics) is instead gapped and tends to  $-i\Gamma$  for low  $\mathbf{k}$ : remarkably, the effective damping rate  $\Gamma$  vanishes when the critical point is approached from above, while it tends to the bare damping rate  $\gamma$  far above the threshold.

### Bogoliubov theory of linearized fluctuations. –

The linearized field equations (6) are the starting point to study the fluctuations of the Bose field around the pure condensate solution. As usual, this approach is accurate when fluctuations are small, that is far away from the critical point. A more complete description of the large fluctuations in the critical region requires more sophisticated approaches and will be postponed to further work. A first remarkable work in this direction making use of renormalization group techniques appeared very recently in [23].

Including back the noise term into their RHS, one obtains for each  $\mathbf{k}$  a pair of linear Ito stochastic differential equations from which it is straightforward to compute all the steady-state correlations of the field, e.g. the momentum distribution of the particles

$$n_{\mathbf{k}}^{ss} = (2\pi)^d |\phi_0|^2 \delta^{(d)}(\mathbf{k}) + \frac{D_{\phi\phi}}{\Gamma} \left[ \frac{\mu^2 + \Gamma^2}{E_{\mathbf{k}}^2} + 1 \right] : \quad (8)$$

in experiments, this quantity is directly accessible as the angular distribution of the far field emission. The former term in (8) describes the condensate, while the latter one account for the non-condensed cloud. Note that the UV divergence due to the second term in the bracket (the one with no  $\mathbf{k}$  dependence) is an artifact of the white bath assumption and disappears as soon as the frequency dependence of the bath is included in the theory.

In what follows, we shall instead concentrate on the low- $\mathbf{k}$ , long-distance physics, where our effective theory is accurate. The low- $\mathbf{k}$  behavior of  $n_{\mathbf{k}}^{ss}$  in the different cases is summarized in the table and compared to the prediction of the standard Bogoliubov theory [24] for equilibrium systems at  $T = 0$  and at  $T > 0$ .

	Equil. $T = 0$	Equil. $T \neq 0$	Non-Equil.
$g \neq 0$	$\frac{\sqrt{m\mu}}{2k}$	$\frac{mT}{k^2}$	$\frac{mD_{\phi\phi}(\mu^2 + \Gamma^2)}{\mu\Gamma} \frac{1}{k^2}$
$g = 0$	-	$\frac{2mT}{k^2}$	$4mD_{\phi\phi}\Gamma \frac{1}{k^4}$

In the presence of interactions  $g > 0$ , the behavior of the non-equilibrium system recovers the one of the corresponding equilibrium system at an effective temperature

$$T_{eff} = \frac{D_{\phi\phi}(\mu^2 + \Gamma^2)}{\mu\Gamma} : \quad (9)$$

independently of the absence of thermalizing collisions between different Bogoliubov modes, the momentum distribution recovers the typical shape of a gas at thermal equilibrium,  $n_{\mathbf{k}}^{ss} \propto k^{-2}$ . As a result, the mere observation of a thermal-like momentum distribution does not appear to be an unambiguous proof of a thermal equilibrium state in the polariton gas. The situation is very different in the case of a non-interacting gas with  $g = 0$ , where the momentum distribution scales as  $n_{\mathbf{k}}^{ss} \propto k^{-4}$ .

From the momentum distribution  $n_{\mathbf{k}}^{ss}$ , it is immediate to obtain by Fourier transform the one-time spatial corre-

lation function

$$G(|\mathbf{x} - \mathbf{y}|) = \langle \phi^*(\mathbf{x}) \phi(\mathbf{y}) \rangle = \int \frac{d^d \mathbf{k}}{(2\pi)^d} n_{\mathbf{k}}^{ss} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}. \quad (10)$$

Long-range order is then defined according to the Penrose-Onsager criterion by the condition

$$\lim_{|\mathbf{x}-\mathbf{y}| \rightarrow \infty} \frac{G(|\mathbf{x} - \mathbf{y}|)}{G(0)} \neq 0. \quad (11)$$

While the UV divergence in (10) is an artifact of our white bath model and can be safely removed by means of a UV cut-off, the convergence vs. divergence of the integral on the IR side determines the presence vs. absence of long-range order. Inserting the explicit formula (8) into (10), one immediately sees that the long-range order in an interacting gas is only stable in  $d \geq 3$ , while an even higher  $d \geq 5$  is required in a non-interacting gas. Once again, the prediction for the interacting gas recovers known results for the equilibrium gas and can be seen as a non-equilibrium version of the Mermin-Wagner theorem of equilibrium statistical mechanics [25].

**Density-phase Bogoliubov theory.** – In low- $d$  where the condensate is destabilized by fluctuations, the Bogoliubov theory sketched in the previous section is inconsistent in the thermodynamic limit, as it implicitly assumes the existence of a condensate to start with. To go beyond this difficulty, more sophisticated versions of the Bogoliubov theory have been developed, based on the idea of a *quasi*-condensate [26,27]: The basic idea is to separate fluctuations in density and phase fluctuations according to

$$\phi(\mathbf{x}, t) = \sqrt{n_0 + \delta n(\mathbf{x}, t)} e^{i\theta(\mathbf{x}, t)} e^{-i\omega t}. \quad (12)$$

We then assume that density fluctuations  $\delta n$  around the average value  $n_0$  are weak  $\delta n \ll n_0$  and that the phase  $\theta$  smoothly varies in space. As the phase  $\theta$  can take arbitrarily large values, no assumption on the presence of a condensate is made and the theory is able to describe configurations with no long-range order.

Within this density-phase Bogoliubov approach, the different  $\mathbf{k}$  components of density and phase fluctuations decouple and evolve according to linear differential equations very similar to (6) including Ito noise terms. Averaging over noise, we obtain the following forms for the correlations of density and phase fluctuations at a given  $\mathbf{k}$ ,

$$\langle \delta n_{\mathbf{k}} \delta n_{\mathbf{k}'} \rangle_{ss} = \delta_{\mathbf{k}\mathbf{k}'} \frac{2n_0 D_{\phi\phi}}{\Gamma} \left( 1 - \frac{\mu}{\epsilon_{\mathbf{k}} + 2\mu} \right) \quad (13)$$

$$\begin{aligned} \langle \theta_{\mathbf{k}} \theta_{\mathbf{k}'} \rangle_{ss} &= \delta_{\mathbf{k}\mathbf{k}'} \frac{D_{\phi\phi}}{2n_0 \Gamma} \left( 1 + \frac{2(\mu^2 + \Gamma^2)}{E_{\mathbf{k}}^2} + \frac{\mu}{\epsilon_{\mathbf{k}} + 2\mu} \right) \end{aligned} \quad (14)$$

$$\langle \delta n_{\mathbf{k}} \theta_{\mathbf{k}'} \rangle_{ss} = \delta_{\mathbf{k}\mathbf{k}'} \frac{D_{\phi\phi}}{\epsilon_{\mathbf{k}} + 2\mu} \quad (15)$$

where  $\delta_{\mathbf{k}\mathbf{k}'} = (2\pi)^d \delta^{(d)}(\mathbf{k} - \mathbf{k}')$ .

**Spatial coherence in reduced  $d$ .** – Fourier transform of (13) back to real space shows that the correlation function of density fluctuations quickly tends to zero on a length scale set by the usual healing length  $\xi = \sqrt{\hbar^2/\mu m}$ . At larger distances, the field correlation functions can then be written in terms of phase fluctuations only. As these inherit the Gaussian statistics of the noise, we can rewrite the field correlation as

$$G(\mathbf{x} - \mathbf{y}) \simeq n_0 e^{-G_{\theta\theta}(|\mathbf{x}-\mathbf{y}|)}, \quad (16)$$

in terms of the phase correlation function

$$G_{\theta\theta}(\mathbf{x} - \mathbf{y}) = \frac{1}{2} \langle [\theta(\mathbf{x}) - \theta(\mathbf{y})]^2 \rangle. \quad (17)$$

Focussing on the long-distance behavior of the spatial correlation function, we can safely neglect the short range correlations due the first term in (14) and write the correlation function in the form

$$\begin{aligned} G_{\theta\theta}(x) &= -\xi^2 \xi_0^{d-4}(\mu) \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\mathbf{k} \cdot \mathbf{x}} - 1}{k^2} + \\ &+ \xi^2 \xi_0^{d-4}(0) \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\mathbf{k} \cdot \mathbf{x}} - 1}{k^2 + \xi^{-2}} \end{aligned} \quad (18)$$

where the additional length scale  $\xi_0$  is defined as

$$\xi_0(\mu) = \left( \frac{4m^2 D_{\phi\phi}(\mu^2 + \Gamma^2)}{n_0 \Gamma} \right)^{\frac{1}{d-4}}. \quad (19)$$

In the following of this Section, we shall separately consider the physically relevant  $d = 1, 2, 3$  cases, and highlight the peculiarities of each of them.

$d = 3$ . As we have already mentioned, in  $d = 3$  both  $G_{\theta\theta}$  and  $G$  correlation functions have finite long-distance limits as soon as the interaction strength is finite  $g > 0$ : this signals the existence of long-range order, while the momentum distribution of the non-condensed particles follows the typical  $k^{-2}$  law of an equilibrium system at the effective temperature  $T_{eff}$  determined by (9). On the other hand, in the opposite case of a non-interacting gas with  $g = 0$ , taking the  $\mu \rightarrow 0$  limit leads to a field correlation function

$$G(x) \propto e^{-\frac{1}{8\pi} \frac{x}{\xi_0(0)}} \quad (20)$$

that exponentially decays at large distances within the characteristic length  $\xi_0$ . This is, of course, in stark contrast with the existence of a BEC transition in the textbook case of a  $d = 3$  ideal Bose gas.

$d = 2$ . In this case, the long-distance behavior of the correlation function can be evaluated making use of known properties of the Bessel functions to have the power-law form

$$G(|\mathbf{x} - \mathbf{y}|) \simeq \text{const} \frac{\xi^\eta}{|\mathbf{x} - \mathbf{y}|^\eta}, \quad (21)$$

with an exponent

$$\eta = \frac{1}{2\pi} \frac{\xi^2}{\xi_0^2(\mu)} = \frac{m D_{\phi\phi}(\mu^2 + \Gamma^2)}{2\pi n_0 \mu \Gamma}. \quad (22)$$



This power law decay behaviour is characteristic of a superfluid (but not Bose-Einstein condensed) two-dimensional Bose gas at temperatures below the Berezinskii-Kosterlitz-Thouless transition [28] and was first discussed in a non-equilibrium context in [14]. Identifying the superfluid density  $n_s$  with the average density  $n_0$  and the temperature  $T$  with the effective temperature  $T_{eff}$  defined in (9), the form (22) of the exponent  $\eta$  matches the usual form  $\eta_{BKT} = 1/\rho_s \lambda_T^2$ , for equilibrium BKT gases, with the de Broglie wavelength defined as usual as  $\lambda_T = [2\pi\hbar^2/mk_B T]^{1/2}$ . In spite of this, it is essential to note that, while in the equilibrium case thermodynamic stability of the quasi-ordered BKT phase imposes an upper bound  $\eta < 1/4$  to the exponent [28], no such condition is known for the non-equilibrium case and larger exponents might be observable. Experimental indications in this direction were recently reported for an interacting polariton gas in [29].

On the other hand, the field correlation function of a non-interacting  $g \rightarrow 0$  gas has the form

$$G_{\theta\theta}(r) \simeq \frac{1}{8\pi} \left[ 1 - \mathcal{G} + \ln 2 - \ln \left( \frac{r}{\xi} \right) \right] \left( \frac{r}{\xi_0(0)} \right)^2, \quad (23)$$

which quadratically grows in the  $r \gg \xi_0$  limit.  $\mathcal{G}$  is here the Euler-Mascheroni constant,  $\mathcal{G} \simeq 0.577$ ; the logarithmic correction remains finite as the  $g = 0$  condition corresponds to a diverging value of the healing length  $\xi$ . As a result, all the quasi-order that was present in the BKT state disappears in the  $g \rightarrow 0$  limit and is replaced by a much faster almost Gaussian decay: experimental results shown in the same article [29] for the very weakly interacting photon regime suggest that in this case the correlation function indeed decays in the quasi-Gaussian way predicted by (23).

$d = 1$ . Straightforward integration of (18) in the  $d = 1$  case predicts a fast, diverging growth of  $G_{\theta\theta}$  of the form

$$G_{\theta\theta}(x) = \frac{1}{2} \left[ \left( \frac{\xi}{\xi_0(\mu)} \right)^3 \frac{x}{\xi} + \left( \frac{\xi}{\xi_0(0)} \right)^3 (e^{-|x|/\xi} - 1) \right]. \quad (24)$$

Depending on the relative value of the two characteristic distances  $\xi$  and  $\xi_0$ , this leads to different functional forms of the long-distance tail of  $G(x)$ .

For  $\xi_0 \gg \xi$  (which is the case of an interacting gas at strong pump values well above threshold), the field correlation function has an exponential tail  $G(x) \simeq e^{-|x|/\ell_1}$  as first predicted in [13] for the OPO pumping scheme. This exponential form closely follow the corresponding case of a  $d = 1$  gas at equilibrium at a finite  $T > 0$  [30]. Also the characteristic length  $\ell_1$  has the same form in the two cases: the explicit expression

$$\ell_1 = \frac{2n_0\mu\Gamma}{mD_{\phi\phi}(\mu^2 + \Gamma^2)} \quad (25)$$

matches the equilibrium one  $\ell_T^D = n\lambda_T^2/\pi$  upon identifying the temperature  $T$  with the effective temperature  $T_{eff}$

defined in (9). The situation is somehow different in a non-interacting gas and, more in general, for  $\xi_0 \ll \xi$ : in this case, the long-distance tail can be approximated with a Gaussian form  $G(x) \simeq e^{-(x/\ell_2)^2}$  of characteristic length  $\ell_2 = 2\sqrt{\xi_0^3(0)/\xi}$ . Experimental indications of a reduced coherence in one-dimensional polariton condensates was reported in [31, 32].

**Frequency dependent pumping.** — All the discussion so far has been based on a model assuming amplification and dissipation baths with no frequency dependence. This simplifying assumption was based on the physical intuition that the correlation time of the bath should be very short, which justifies a Markovian approach based on Ito partial differential equations. More sophisticated kinetic descriptions of the amplification mechanism [33, 34] as well as experimental observations of condensation in harmonic traps [35] suggest the presence of some mechanism tending to concentrate the emission into low-energy modes and quench the high-energy ones.

Following [36], a simple way to model this behavior is to include some frequency-dependence of the amplification on a characteristic scale  $\omega_c$ , e.g. by replacing the amplification coefficient  $P_0$  with the differential operator  $P = P_0 [1 + \omega_c^{-1}(\omega_0 - i\partial/\partial t)]$ : in this way, amplification (and then condensation) into low energy modes is favored, while it is completely suppressed above the cut-off frequency  $\omega > \omega_c + \omega_0$ . As a result, the transition point is still at  $P_0^c = \gamma$ , but the competition of the frequency-dependent amplification with the interaction-induced blue shift makes the condensate density to saturate at large pump powers,  $|\phi|^2 = n_s (P_0 - P_0^c)/(P_0^c + P_0 g n_s / \omega_c)$ .

An example of Bogoliubov excitation spectrum around the spatially homogeneous, plane-wave condensate solution is shown in the right panels of Fig.1: while the Goldstone mode with a diffusive dispersion at low  $\mathbf{k}$ 's is still visible, the damping rate of the Bogoliubov modes rapidly grows at large  $\mathbf{k}$ 's. The momentum distribution of the excitations is obtained by adding phenomenological noise terms to the Bogoliubov equations as done in the first part of this Letter. This leads to the closed expression:

$$n_{\mathbf{k}}^{ss} = \frac{D_{\phi\phi}}{\tilde{\Gamma} + M(\epsilon_{\mathbf{k}} + \mu)} \left( 1 + \frac{\mu^2 + \tilde{\Gamma}^2}{E_{\mathbf{k}}^2} \right), \quad (26)$$

in terms of the parameters  $\tilde{\Gamma} = \gamma(P_0 - P_0^c)/[P_0(1 + g n_s / \omega_c)]$  and  $M = P_0/[\omega_c(1 + |\phi_0|^2/n_s)]$ . While the low- $\mathbf{k}$  part of the momentum distribution (26) retains the same behavior as in (8), the frequency-dependent amplification is able to regularize the UV behavior of the theory, making all relevant integrals to nicely converge at large  $\mathbf{k}$ . Of course a quantitative account of the details of the momentum distribution requires a more sophisticated model of the frequency-dependence, still our model is able to qualitatively recover its fast decrease at large  $\mathbf{k}$  as typically observed in Bose-Einstein condensation experiments. In particular, note how our theory predicts that this behav-

ior is completely independent of the collisional thermalization rate: this may explain the unexpected experimental observation of a thermal-like distribution in a gas of non-interacting photons in a VCSEL device in the weak-coupling regime [37]. The role of this effect in determining the thermal photon distribution observed in the photon BEC experiment of [12] is under investigation.

**Conclusions.** — In this Letter we have developed a fully analytical theory of the coherence properties of non-equilibrium Bose-Einstein quasi-condensates in reduced dimensionalities. Even though the non-equilibrium stationary state is determined by a dynamical balance of pumping and dissipation rather than by a standard thermodynamical equilibrium condition, the momentum distribution and the long-distance spatial coherence of a weakly interacting gas is qualitatively similar to the one of the corresponding equilibrium system at a finite temperature. Most remarkably, this conclusion does not depend on the collisional thermalization rate in the gas and, with some modifications, extends to the case of non-interacting particles. Thanks to the generic nature of our theory, its conclusions can be applied to a broad range of systems displaying a non-equilibrium BEC-like phase transition, from laser operation in VCSEL devices, to photon BECs, to polariton BECs. As a next step, we will derive explicit predictions for the amplification and noise parameters starting from microscopic models of the most relevant physical systems, so to provide quantitative predictions for the emission spectra and the coherence functions in the different cases.

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